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Axiomatizations of Pareto Equilibria in Multicriteria Games

Mark Voorneveld[1]

Dries Vermeulen[2]

Peter Borm[3]

Abstract: We focus on axiomatizations of the Pareto equilibrium concept in multicriteria games based on consistency. Axiomatizations of the Nash equilibrium concept by Peleg and Tijs (1996) and Peleg, Potters, and Tijs (1996) have immediate generalizations. The axiomatization of Norde *et al.* (1996) cannot be generalized without the use of an additional axiom.

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[1]Corresponding author; Department of Econometrics and CentER, Tilburg University, P.O.Box 90153, 5000 LE Tilburg, The Netherlands, M.Voorneveld@kub.nl

[2]Department of Quantitative Economics, Maastricht University, P.O.Box 616, 6200 MD Maastricht, The Netherlands

[3]Department of Econometrics and CentER, Tilburg University, P.O.Box 90153, 5000 LE Tilburg, The Netherlands

1 Introduction

In a recent manifesto Bouyssou *et al.* (1993) observe that within multicriteria decision making '[a] systematic axiomatic analysis of decision procedures and algorithms is yet to be carried out'. In this paper, several axiomatizations of the Pareto equilibrium concept for multicriteria games are provided.

In multicriteria games, a player can have more than one criterion function. Shapley (1959) introduced (Pareto) equilibrium points for this type of games that are a straightforward generalization of the Nash equilibrium concept for unicriterion games.

Axiomatic properties of the Nash equilibrium concept based on the notion of consistency have been studied in several articles, including Peleg and Tijs (1996), Peleg, Potters, and Tijs (1996), and Norde *et al.* (1996). Several of these axiomatizations carry over to multicriteria games. However, we prove that the strong result of Norde *et al.* (1996), characterizing the Nash equilibrium concept on the set of mixed extensions of finite strategic form games by nonemptiness, the selection of all utility maximizing strategies in one person games, and consistency in the sense that equilibrium strategies in a game must remain equilibrium strategies if players know that others will play their equilibrium strategies, does not have such an analogon in multicriteria games. In this note we show that nonemptiness, consistency and an immediate extension of one person rationality are not sufficient to axiomatize the Pareto equilibrium concept. An additional property is provided to establish an axiomatization.

2 Preliminaries

A finite multicriteria game is a tuple $G = \langle N^G, (X_i)_{i\in N^G}, (u_i)_{i\in N^G}\rangle$, where $N^G \subset \mathbb{N}$ is a finite set of players, X_i is the finite set of pure strategies of player $i \in N^G$, and for each player $i \in N^G$, the function $u_i : \prod_{i\in N^G} X_i \to \mathbb{R}^{r(i)}$ maps each strategy combination to a point in $r(i)$-dimensional Euclidean space. The interpretation of this last function is that player $i \in N^G$ considers not just one, but $r(i) \in \mathbb{N}$ different criteria. For notational convenience, the set of finite multicriteria games is denoted Γ_{finite}.

The payoff functions are extended to mixed strategies in the obvious way. The set of mixed extensions of finite multicriteria games is denoted by Γ. This set contains the set $\Gamma_{strategic}$ of mixed extensions of finite games in strategic form, since these are simply multicriteria games in which each player has only one criterion.

For notational convenience, let $\Delta(X)$ denote the set of probability measures on a finite set X, and for $m \in \mathbb{N}$, $\Delta_m := \{\mu \in \mathbb{R}^m_+ | \sum_{i=1}^m \mu_i = 1\}$ is the unit simplex in $\mathbb{R}^m$.

Let $G = \langle N^G, (X_i)_{i\in N^G}, (u_i)_{i\in N^G}\rangle \in \Gamma$ be a multicriteria game, let $x \in \prod_{i\in N^G} \Delta(X_i)$ be a strategy profile in G, and let $S \subset N^G$ be a proper subcoalition of the player set N^G, i.e., $S \neq \emptyset, N^G$. The *reduced game* $G^{S,x}$ of G with respect to S and x is the multicriteria game in which

- the player set is S;
- each player $i \in S$ has the same set X_i of pure strategies as in G;
- the payoff functions $(u'_i)_{i\in S}$ are defined by $u'_i(y_S) := u_i(y_S, x_{N^G\setminus S})$ for all $y_S \in \prod_{i\in S} \Delta(X_i)$.

A solution concept on Γ is a function ϕ which assigns to each element $G \in \Gamma$ a subset $\phi(G) \subseteq \prod_{i \in N^G} \Delta(X_i)$ of strategy combinations. Analogously one defines a solution concept on $\Gamma_{strategic}$ or Γ_{finite}.

For strategic form games, we recall the following axioms. A solution concept ϕ on $\Gamma_{strategic}$ satisfies:

- **Nonemptiness (NEM)**, if $\phi(G) \neq \emptyset$ for all $G \in \Gamma_{strategic}$;
- **One Person Rationality (OPR)**, if for each $G \in \Gamma_{strategic}$ with $|N^G| = 1$ we have that $\phi(G) = \{x \in \Delta(X_i) | u_i(x) \geq u_i(y) \; \forall y \in \Delta(X_i)\}$, the set of utility maximizing strategies;
- **Consistency (CONS)**, if for each game $G \in \Gamma_{strategic}$, each proper subcoalition $S \subset N^G$, and each element $x \in \phi(G)$, we have that $x_S \in \phi(G^{S,x})$.

Norde *et al.* (1996) prove:

Proposition 2.1 *A solution concept ϕ on $\Gamma_{strategic}$ satisfies NEM, OPR, and CONS if and only if $\phi = NE$, the Nash equilibrium concept.*

Shapley (1959) introduces equilibrium points for multicriteria games. Let $G \in \Gamma$ be a multicriteria game. A *Pareto equilibrium* is a strategy combination $x \in \prod_{i \in N^G} \Delta(X_i)$ such that for each $i \in N^G$, there does not exist a $\widetilde{x}_i \in \Delta(X_i)$ such that:

$$u_i(\widetilde{x}_i, x_{-i}) > u_i(x_i, x_{-i}),$$

where for two vectors $a, b \in \mathbb{R}^m$, we write $a > b$ if $a_i > b_i$ for all $i = 1, \ldots, m$. The solution concept on Γ assigning to each $G \in \Gamma$ the set of Pareto equilibria is denoted by PE. The Pareto equilibrium concept PE ON Γ_{finite} is, of course, defined in a similar way by restricting attention to pure, rather than mixed, strategies.

Consider a multicriteria game $G \in \Gamma$ in which player i has $r(i) \in \mathbb{N}$ criteria. For each $i \in N^G$, let $\lambda_i \in \Delta_{r(i)}$ be a vector of weights for the criteria, $\lambda := (\lambda_i)_{i \in N^G}$. The *$\lambda$-weighted game* G_λ is the strategic form game with player set N^G, mixed strategy spaces $(\Delta(X_i))_{i \in N^G}$, and payoff functions $(v_i)_{i \in N^G}$ defined for all $i \in N^G$ and $x \in \prod_{i \in N^G} \Delta(X_i)$ by $v_i(x) = \langle \lambda_i, u_i(x) \rangle = \sum_{k=1}^{r(i)} \lambda_{ik} u_{ik}(x)$. Shapley (1959) proves that all Pareto equilibria can be found by a suitable weighing of the criteria of the players:

Lemma 2.2 *For each $G \in \Gamma$: $x \in PE(G)$ if and only if there exists for each $i \in N^G$ a vector of weights $\lambda_i \in \Delta_{r(i)}$ such that $x \in NE(G_\lambda)$.*

As a corollary, Pareto equilibria always exist in mixed extensions of finite multicriteria games, since for any vector of weights the game G_λ has Nash equilibria in mixed strategies.

3 Finite Multicriteria Games

Peleg and Tijs (1996) and Peleg, Potters, and Tijs (1996) provide several axiomatizations of the Nash equilibrium concept for finite strategic form games. In this section two of these axiomatizations are extended in a straightforward manner to finite multicriteria games.

We use the following axioms. A solution concept ϕ on Γ_{finite} satisfies:

- **Restricted Nonemptiness (r-NEM)**, if for every $G \in \Gamma_{finite}$ with $PE(G) \neq \emptyset$ we have $\phi(G) \neq \emptyset$;
- **One Person Efficiency (OPE)**, if for each $G \in \Gamma_{finite}$ with $|N^G| = 1$ we have that $\phi(G) = \{x \in X_i | \nexists y \in X_i : u_i(y) > u_i(x)\}$, the set of strategies that yield a maximal payoff w.r.t. $>$ on $\mathbb{R}^{r(i)}$;
- **Consistency (CONS)**, if for each $G \in \Gamma_{finite}$, each proper subcoalition $S \subset N^G$, and each element $x \in \phi(G)$, we have that $x_S \in \phi(G^{S,x})$;
- **Converse Consistency (COCONS)**, if for each $G \in \Gamma_{finite}$ with at least two players, we have that $\widetilde{\phi}(G) \subseteq \phi(G)$, where

$$\widetilde{\phi}(G) = \{x \in \prod_{i \in N^G} X_i | \forall S \in 2^{N^G} \setminus \{\emptyset, N^G\} : x_S \in \phi(G^{S,x})\}.$$

According to restricted nonemptiness, the solution concept provides a nonempty set of strategies whenever Pareto equilibria exist. One person efficiency claims that in games with only one person, the solution concept picks out all strategies which yield a maximal payoff with respect to the $>$ order of $\mathbb{R}^{r(i)}$. Consistency means that a solution x of a game is also a solution of each reduced game in which the players that leave the game play according to the strategies in x. Converse consistency prescribes that a strategy combination which gives rise to a solution in every reduced game is also a solution of the original game.

Notice that consistency can be defined by the opposite inclusion as in converse consistency, namely $\phi(G) \subseteq \widetilde{\phi}(G)$ for every $G \in \Gamma_{finite}$.

Our first result indicates that the axiomatization of the Nash equilibrium concept on finite strategic games of Peleg, Potters, and Tijs (1996, Thm. 3) in terms of restricted nonemptiness, one person rationality and consistency can be generalized to multicriteria games.

Theorem 3.1 *A solution concept ϕ on Γ_{finite} satisfies r-NEM, OPE, and CONS if and only if $\phi = PE$.*

Proof. It is clear that PE satisfies the axioms. Let ϕ be a solution concept on Γ_{finite} satisfying r-NEM, OPE, and CONS. Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma_{finite}$. We first show that $\phi(G) \subseteq PE(G)$. If $\phi(G) = \emptyset$, we are done. So let $x \in \phi(G)$. Then $x_i \in \phi(G^{i,x})$ by CONS, so $x_i \in \{y_i \in X_i | \nexists z_i \in X_i : u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})\}$ by OPE. Hence x is a Pareto equilibrium: $x \in PE(G)$. To prove the converse, i.e., that $PE(G) \subseteq \phi(G)$, consider $\hat{x} \in PE(G)$. Construct a finite multicriteria game H as follows:

- the player set is $N^G \cup \{0\}$;
- players $i \in N^G$ have the same strategy set X_i as in G;
- player 0 has strategy set $\{0, 1\}$;
- payoff functions v_i to players $i \in N^G$ are defined, for all $(x_0, x) \in \{0, 1\} \times \prod_{i \in N^G} X_i$, by:

$$v_i(x_0, x) = \begin{cases} u_i(x) & \text{if} \quad x_0 = 1 \\ -e^{r(i)} & \text{if} \quad x_0 = 0, x_i \neq \hat{x}_i \\ e^{r(i)} & \text{if} \quad x_0 = 0, x_i = \hat{x}_i \end{cases}$$

where $e^{r(i)} \in \mathbb{R}^{r(i)}$ is the vector with each component equal to one.

- the payoff function v_0 to player 0 is defined, for all $(x_0, x) \in \{0,1\} \times \prod_{i \in N^G} X_i$, by:

$$v_0(x_0, x) = \begin{cases} 0 & \text{if} \quad x_0 = 0 \\ -1 & \text{if} \quad x_0 = 1, x \neq \hat{x} \\ 1 & \text{if} \quad x_0 = 1, x = \hat{x} \end{cases}$$

Simple verification indicates that $(1, \hat{x})$ is the unique Pareto equilibrium of H. Since $\phi(H) \subseteq PE(H)$, we conclude by r-NEM that $(1, \hat{x}) \in \phi(H)$. Then by CONS, $\hat{x} \in \phi(H^{N^G,(1,\hat{x})}) = \phi(G)$, since by definition $H^{N^G,(1,\hat{x})} = G$. Hence $\hat{x} \in \phi(G)$, finishing our proof. $\square$

Our second result shows that the axiomatization of the Nash equilibrium concept on finite strategic games of Peleg and Tijs (1996, Thm. 2.12) in terms of one person rationality, consistency and converse consistency can also be generalized to multicriteria games.

Theorem 3.2 *A solution concept ϕ on Γ_{finite} satisfies OPE, CONS, and COCONS if and only if $\phi = PE$.*

Proof. As in the proof of Theorem 3.1, we have that $\phi \subseteq PE$ by OPE and CONS. To prove that $PE \subseteq \phi$, we use induction on the number of players. In one player games, the claim follows from OPE. Now assume the claim holds for all finite multicriteria games with at most n players and let $G \in \Gamma_{finite}$ be an $(n+1)$-player game. By CONS: $PE(G) \subseteq \widetilde{PE}(G)$. By induction: $\widetilde{PE}(G) \subseteq \widetilde{\phi}(G)$. By COCONS: $\widetilde{\phi}(G) \subseteq \phi(G)$. Combining these three inclusions: $PE(G) \subseteq \phi(G)$. $\square$

These results seem to illustrate that the axiomatizations that exist in the literature for the Nash equilibrium concept generalize in a straightforward manner to the Pareto equilibrium concept in multicriteria games. This analogy, however, breaks down when we consider mixed extensions of finite multicriteria games, as is done in the next section.

4 Mixed Extensions of Finite Multicriteria Games

Norde *et al.* (1996) characterize the Nash equilibrium concept on mixed extensions of finite strategic form games by nonemptiness, one person optimality, and consistency. In this section it is shown that analogons of these properties are not sufficient to characterize the Pareto equilibrium concept in mixed extensions of finite multicriteria games.

First, we list some of the axioms used in this section. A solution concept ϕ on Γ satisfies:

- **Nonemptiness (NEM)**, if $\phi(G) \neq \emptyset$ for each $G \in \Gamma$;

- **One Person Efficiency (OPE)**, if for each $G \in \Gamma$ with $|N^G| = 1$ we have that $\phi(G) = \{x \in \Delta(X_i) | \nexists y \in \Delta(X_i) : u_i(y) > u_i(x)\}$, the set of strategies that yield a maximal payoff w.r.t. $>$ on $\mathbb{R}^{r(i)}$;

- **Consistency (CONS)**, if for each $G \in \Gamma$, each proper subcoalition $S \subset N^G$, and each element $x \in \phi(G)$, we have that $x_S \in \phi(G^{S,x})$;

- **Converse Consistency (COCONS)**, if for each $G \in \Gamma$ with at least two players, we have that $\widetilde{\phi}(G) \subseteq \phi(G)$, where

$$\widetilde{\phi}(G) = \{x \in \prod_{i \in N^G} \Delta(X_i) | \forall S \in 2^{N^G} \setminus \{\emptyset, N^G\} : x_S \in \phi(G^{S,x})\}.$$

It is easy to see that PE on Γ satisfies NEM (See Lemma 2.2), OPE, and CONS. Moreover,

Lemma 4.1 *If a solution concept ϕ on Γ satisfies OPE and CONS, then $\phi \subseteq PE$.*

Proof. Let ϕ be a solution concept on Γ, satisfying OPE and CONS. Let $G \in \Gamma$. If $\phi(G) = \emptyset$, we are done. So now let $x \in \phi(G)$. Then $x_i \in \phi(G^{i,x})$ by CONS, so $x_i \in \{y_i \in \Delta(X_i) | \nexists z_i \in \Delta(X_i) : u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})\}$ by OPE. Hence x is a Pareto equilibrium: $x \in PE(G)$. □

Obviously, PE is the largest solution concept on Γ satisfying NEM, OPE, and CONS, but not the only one, as our next result shows.

Proposition 4.2 *There exists a solution concept ϕ on Γ which satisfies NEM, OPE, and CONS, such that $\phi \neq PE$.*

Proof. Define ϕ as follows. Let $G = \langle N^G, (X_i)_{i \in N^G}, (u_i)_{i \in N^G} \rangle \in \Gamma$.

- If $|N^G| = 1$, take $\phi(G) = \{x \in \Delta(X_i) | \nexists y \in \Delta(X_i) : u_i(y) > u_i(x)\}$, the nondominated strategies as required in OPE.
- If $|N^G| > 1$, take

  $$\phi(G) := \{x \in \prod_{i \in N^G} \Delta(X_i) | \forall i \in N^G : \nexists y_i \in \Delta(X_i) \text{ such that } u_i(y_i, x_{-i}) \geq u_i(x_i, x_{-i})\},$$

  where for $a, b \in \mathbb{R}^m$ we write $a \geq b$ if $a_i \geq b_i$ for all $i = 1, \ldots, m$ and $a \neq b$. Shapley (1959) calls this the set of strong equilibrium points and provides an existence theorem similar to Lemma 2.2, thereby establishing NEM.

It is easy to see that ϕ is also consistent. To show that $\phi \neq PE$, consider the game G in Figure 1, where the first player has two and the second player only one criterion. Obviously $(B, L) \in PE(G)$, but $(B, L) \notin \phi(G)$, since $u_1(T, L) \geq u_1(B, L)$. □

	L	R
T	(1,1),0	(1,1),0
B	(1,0),0	(1,0),0

Figure 1.

In order to arrive at an axiomatization of PE, we require an additional axiom. A solution concept ϕ on Γ satisfies:

- **WEIGHT** if for every game $G \in \Gamma$ and each vector $(\lambda_i)_{i \in N^G} \in \prod_{i \in N^G} \Delta_{r(i)}$ of weights: $\phi(G_\lambda) \subseteq \phi(G)$.

In other words, ϕ satisfies WEIGHT if some of the solutions of a multicriteria game may be found by weighing the criteria and computing the solution in the weighted strategic form game.

Our main result, using the strong theorems of Norde *et al.* (1996) and Shapley (1959), shows that the Pareto equilibrium concept is the unique solution concept on Γ satisfying NEM, OPE, CONS, and WEIGHT.

Theorem 4.3 *A solution concept ϕ on Γ satisfies NEM, OPE, CONS, and WEIGHT if and only if $\phi = PE$.*

Proof. Straightforward verification and application of Lemma 2.2 indicates that PE indeed satisfies the four axioms. Now let ϕ be a solution concept on Γ satisfying NEM, OPE, CONS, and WEIGHT. By Lemma 4.1, $\phi \subseteq PE$. Now let $G \in \Gamma$, and $x \in PE(G)$, which is possible by NEM. Remains to show that $x \in \phi(G)$. By Lemma 2.2, there exists a vector $\lambda = (\lambda_i)_{i \in N^G} \in \prod_{i \in N^G} \Delta_{r(i)}$ of weights such that $x \in NE(G_\lambda)$. Notice that ϕ restricted to $\Gamma_{strategic}$, the set of mixed extensions of strategic form games, satisfies NEM, OPR, and CONS, and hence by Proposition 2.1, $\phi(H) = NE(H)$ for all $H \in \Gamma_{strategic}$. Consequently, $\phi(G_\lambda) = NE(G_\lambda) \ni x$. So by WEIGHT: $x \in \phi(G)$. $\square$

Remark 4.4 The strong Pareto equilibrium concept, as mentioned in the proof of Proposition 4.2, can be axiomatized analogously by nonemptiness, consistency, one person strong efficiency, and a weight axiom concerning strictly positive, rather than nonnegative, weights.

Finally, without proof, we mention that the analogon of Theorem 3.2 also holds when we consider mixed extensions:

Theorem 4.5 *A solution concept ϕ on Γ satisfies OPE, CONS, and COCONS if and only if $\phi = PE$.*

It is an easy exercise to show that the axioms used in our theorems are logically independent.

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